

# GENERALIZED ALGEBRAIC RATIONAL IDENTITIES OF SUBNORMAL SUBGROUPS IN DIVISION RINGS

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**ABSTRACT.** In this note, we introduce a new concept of a *generalized algebraic rational identity* to investigate the structure of division rings. The main theorem asserts that if a non-central subnormal subgroup  $N$  of the multiplicative group  $D^*$  of a division ring  $D$  with center  $F$  satisfies a non-trivial generalized algebraic rational identity of bounded degree, then  $D$  is a finite dimensional vector space over  $F$ . This generalizes some previous results.

Let  $F$  be a field and  $A$  be an  $F$ -algebra. If  $A$  is finite dimensional over  $F$ , then  $A$  is both algebraic and finitely generated as  $F$ -algebra. In 1941, Kurosh posed a famous problem ([15, Problem R]) by asking that whether  $A$  is finite dimensional provided  $A$  is algebraic and finitely generated over  $F$ . It is well-known that the Kurosh Problem was solved negatively by Golod and Shafarevich in [10]: they have constructed an example of an infinite dimensional finitely generated algebraic algebra. However, the particular case when  $A$  is a division ring remains unsolved up to present: we do not know whether there exists a division ring algebraic over the center and finitely generated which is infinite dimensional. This case is usually referred as the Kurosh Problem for division rings [15, Problem K]. Now, it is natural to ask what properties whose occurrence in an algebra  $A$  would lead  $A$  to be finite dimensional over its center. In [14], Kaplansky proved that every primitive algebraic algebra of bounded degree is finite dimensional (see [14, Theorem 4]). Recently, Bell *et al.* considered the left algebraicity and they proved the analogue theorem for division rings with this property [3]. In [1], it was proved that if in a division ring  $D$  with center  $F$  all additive commutators  $xy - yx$  are algebraic over  $F$  of bounded degree, then  $D$  is finite dimensional. An analogue result for multiplicative commutators was also obtained in [7]. Namely,  $D$  is finite dimensional in case when either  $xyx^{-1}y^{-1}$  are algebraic over  $F$  of bounded degree for all  $x, y \in D^*$  or  $\text{char}(D) = 0$  and there exists a non-central element  $a$  such that  $axa^{-1}x^{-1}$  are algebraic over  $F$  of bounded degree for all  $x \in D$ . Further, Markar-Limanov, Chiba [8, 17] and some other authors study more general problem. In fact, they study the properties whose occurrence in a subnormal subgroup of  $D^*$  entails  $D$  to be finite dimensional over  $F$ . In particular, it was proved in [8, 17] that if  $F$  is infinite and  $D$  contains a non-central subnormal subgroup satisfying some generalized rational identities, then  $D$  is finite dimensional over  $F$ .

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The aim of the present note is to give a property which would include altogether the properties given for subgroups in the results mentioned above. This idea leads us to the concept of *generalized algebraic rational identities* in a non-central subnormal subgroup of  $D^*$ . From the main result we get in Theorem 8 below, one can easily deduce several previous well-known results.

Let  $D$  be a division ring with center  $F$  and  $F\langle X \rangle$  be a free  $F$ -algebra on a finite set  $X = \{x_1, x_2, \dots, x_m\}$  of non-commuting indeterminates  $x_1, x_2, \dots, x_m$ . We denote by  $D\langle X \rangle = D *_F F\langle X \rangle$  the free product of  $D$  and  $F\langle X \rangle$  over  $F$  and by  $D(X)$  the universal division ring of fractions of  $D\langle X \rangle$ . We call an element  $f(x_1, x_2, \dots, x_m) \in D(X)$  a *generalized rational expression* (or *generalized rational polynomial*) over  $D$ . We refer to [2, Chapter 7] and [19, Chapter 8] for more details on generalized rational polynomials.

Let  $S$  be a subset of  $D$  and  $f(x_1, x_2, \dots, x_m)$  be a generalized rational polynomial over  $D$ . We say that  $f$  is a *generalized algebraic rational identity* (briefly, *GARI*) of  $S$  (or  $S$  satisfies the *GARI*  $f$ ) if  $f(c_1, c_2, \dots, c_m)$  is algebraic over  $F$  whenever  $f$  is defined at  $(c_1, c_2, \dots, c_m) \in S^m$ . A GARI  $f$  of  $S$  is called *non-trivial* if there exists a division ring  $D_1$  with center  $F$  such that  $D_1$  contains all coefficients of  $f$  and  $f$  is not a GARI of  $D_1$ . We say that  $f(x_1, x_2, \dots, x_m)$  is a *generalized rational identity* (briefly, *GRI*) of  $S$  over  $D$  (or  $S$  satisfies *GRI*  $f = 0$ ) if  $f(c_1, c_2, \dots, c_m) = 0$  whenever  $f$  is defined at  $(c_1, c_2, \dots, c_m) \in S^m$ . Clearly, if  $f$  is a GRI of  $S$ , then  $f$  is a GARI of  $S$ . A GRI  $f$  of  $S$  over  $D$  is called *non-trivial* if there exists a division ring  $D_1$  containing all coefficients of  $f$  such that  $f$  is not a GRI of  $D_1$ . Observe that in a division ring  $D$  which is infinite dimensional over its center, if  $S$  is a non-central subnormal subgroup of the multiplicative group  $D^*$  of  $D$ , then there exists  $(c_1, c_2, \dots, c_m) \in S^m$  such that  $f$  is defined at  $(c_1, c_2, \dots, c_m)$  [8].

For a given positive integer  $n$ , let  $x, y_1, \dots, y_n$  be  $n+1$  non-commuting indeterminates. Consider the following generalized rational expression

$$g_n(x, y_1, y_2, \dots, y_n) = \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) x^{\sigma(0)} y_1 x^{\sigma(1)} y_2 x^{\sigma(2)} \dots y_n x^{\sigma(n)},$$

where  $S_{n+1}$  is the symmetric group defined on the set  $\{0, 1, \dots, n\}$ . The following result is standard and describes algebraic elements of bounded degree.

**Lemma 1.** *Let  $D$  be a division ring with center  $F$ . For any element  $a \in D$ , the following statements are equivalent:*

- (1) *The element  $a$  is algebraic over  $F$  of degree  $\leq n$ .*
- (2)  *$g_n(a, y_1, y_2, \dots, y_n) = 0$  is a GRI on  $D$ .*

*Proof.* See [2, Corollary 2.3.8]. □

Let  $D$  be a division ring with center  $F$  and  $\phi$  be a ring automorphism of  $D$ . We write  $D((\lambda, \phi))$  for the ring of skew Laurent series  $\sum_{i=n}^{\infty} a_i \lambda^i$ , where  $n \in \mathbb{Z}, a_i \in D$ , with the multiplication defined by the twist equation  $\lambda a = \phi(a) \lambda$  for every  $a \in D$ . If  $\phi = Id_D$ , then we write  $D((\lambda))$  instead of  $D((\lambda, Id_D))$ . It is known that  $D((\lambda, \phi))$  is a division ring (see [16, Example 1.8]). Moreover, we have the following results.

**Lemma 2.** [5, Lemma 2.1] *Let  $D$  be a division ring with center  $F$ . Assume that  $K = \{a \in D \mid \phi(a) = a\}$  is the fixed division subring of  $\phi$  in  $D$ . If the center  $k = Z(K)$  of  $K$  is contained in  $F$ , then the center of  $D((\lambda, \phi))$  is*

$$Z(D((\lambda, \phi))) = \begin{cases} k & \text{if } \phi \text{ has infinite order,} \\ k((\lambda^s)) & \text{if } \phi \text{ has a finite order } s. \end{cases}$$

In particular, the center of  $D((\lambda))$  is  $F((\lambda))$ .

**Lemma 3.** *Let  $D$  be a division ring with center  $F$ . An element  $\alpha = a_1\lambda + a_2\lambda^2 + \dots$  in  $D((\lambda))$  is algebraic over  $F$  if and only if  $\alpha = 0$ .*

*Proof.* Suppose that  $\alpha \neq 0$  and  $g(x) = t_0 + t_1x + \dots + t_nx^n \in F[x]$  is the minimal polynomial of  $\alpha$  over  $F$ . Then, the equality

$$0 = t_0 + t_1(a_1\lambda + a_2\lambda^2 + \dots) + t_2(a_1\lambda + a_2\lambda^2 + \dots)^2 + \dots + t_n(a_1\lambda + a_2\lambda^2 + \dots)^n$$

implies  $t_0 = 0$ , that is impossible.  $\square$

For a division ring  $D$  with the center  $F$ , let us consider a countable set of indeterminates  $\{\lambda_i \mid i \in \mathbb{Z}\}$  and a family of division rings which is constructed by setting

$$\begin{aligned} D_0 &= D((\lambda_0)), D_1 = D_0((\lambda_1)), \\ D_{-1} &= D_1((\lambda_{-1})), D_2 = D_{-1}((\lambda_2)), \end{aligned}$$

for any  $n > 1$ ,

$$D_{-n} = D_n((\lambda_{-n})), D_{n+1} = D_{-n}((\lambda_{n+1})).$$

Clearly,  $D_\infty = \bigcup_{n=-\infty}^{+\infty} D_n$  is a division ring. By Lemma 2, it is not hard to prove by induction on  $n \geq 0$  that the center of  $D_0$  is  $F_0 = F((\lambda_0))$ , the center of  $D_{n+1}$  is  $F_{n+1} = F_{-n}((\lambda_{n+1}))$  and the center of  $D_{-n}$  is  $F_{-(n+1)} = F_{n+1}((\lambda_{-(n+1)}))$ . In particular,  $F$  is contained in  $Z(D_\infty)$ . Consider the map  $f : D_\infty \rightarrow D_\infty$  which is defined by  $f(a) = a$  for any  $a \in D$  and  $f(\lambda_i) = \lambda_{i+1}$  for any  $i \in \mathbb{Z}$  is an automorphism of  $D_\infty$ .

**Proposition 4.** *The center of  $D_\infty((\lambda, f))$  is  $F$ .*

*Proof.* We note that  $D$  is the fixed division ring of  $f$  in  $D_\infty$ . Since  $F$  is contained in the center of  $D_\infty$ , the automorphism  $f$  has infinite order. By Lemma 2,  $Z(D_\infty((\lambda, f))) = F$ .  $\square$

**Theorem 5.** *Let  $D$  be a division ring with center  $F$  and  $S$  be a subset of  $D$ . Assume that  $f(x_1, x_2, \dots, x_m) \in D(x_1, x_2, \dots, x_m) \setminus D$  is a GARI of  $S$ . If  $f(c_1, c_2, \dots, c_m) \in F$  for some  $(c_1, c_2, \dots, c_m) \in S^m$ , then  $f$  is a non-trivial GARI.*

*Proof.* It suffices to find a division ring  $L$  containing  $D$  such that  $f$  is not a GARI of  $L$ . Let  $K = D(y_1, y_2, \dots, y_m)$  and  $L = K_\infty((\lambda, f))$ . By Proposition 4,  $Z(L) = Z(K) = Z(D) = F$ . Consider the division subring  $K((\lambda))$  of  $L$ . By Lemma 2,  $F((\lambda))$  is the center of  $K((\lambda))$ . In view of [9, Lemma 7],

$$f(1 + y_1\lambda, 1 + y_2\lambda, \dots, 1 + y_m\lambda) = f(c_1, c_2, \dots, c_m) + \sum_{j=1}^{\infty} f_j(y_1, y_2, \dots, y_m)\lambda^j,$$

where  $f_j(y_1, y_2, \dots, y_m)$  are generalized polynomials over  $D$  and there is  $j_0$  such that  $f_{j_0}(y_1, y_2, \dots, y_m) \neq 0$ . Since  $f(c_1, c_2, \dots, c_m) \in F$ , if

$$f(1 + y_1\lambda, 1 + y_2\lambda, \dots, 1 + y_m\lambda)$$

is algebraic over  $F$ , then  $\sum_{j=1}^{\infty} f_j(y_1, y_2, \dots, y_m) \lambda^j$  is algebraic over  $F$  too. By Lemma 3,  $f_j(y_1, y_2, \dots, y_m) \equiv 0$  for every  $j \geq 1$ . In particular, we have

$$f_{j_0}(y_1, y_2, \dots, y_m) \equiv 0,$$

a contradiction. Thus,  $f(1 + y_1 \lambda, 1 + y_2 \lambda, \dots, 1 + y_m \lambda)$  is not algebraic over  $F$ . Therefore,  $f$  is not a GARI of  $L$ .  $\square$

Recall that for a division ring  $D$  with center  $F$ , an element  $f \in D(X)$  is called a *generalized power central rational identity* (shortly, GPCRI) of a subset  $S$  of  $D$  if  $f$  satisfies the following condition: if  $f$  is defined at  $(c_1, c_2, \dots, c_m) \in S^m$ , then  $f(c_1, c_2, \dots, c_m)^p \in F$  for some positive integer  $p$  (see [9]). Moreover, if  $f^p \notin F$  for any positive integer  $p$ , then we say that  $S$  satisfies a non-trivial GPCRI  $f$ .

**Corollary 6.** *Let  $D$  be a division ring with center  $F$  and  $f(x_1, x_2, \dots, x_m)$  be a generalized rational polynomial over  $D$ . Assume that  $S$  is a subset of  $D$  such that  $f$  is defined at least at an  $m$ -tuple  $(c_1, c_2, \dots, c_m) \in S^m$ . If  $f$  is a non-trivial GPCRI of  $S$ , then  $f$  is a non-trivial GARI of  $S$ .*

*Proof.* Assume that  $f$  is a non-trivial GPCRI of  $S$ . Then, it is obviously that  $f$  is a GARI of  $S$ . Now we will show that  $f$  is a non-trivial GARI of  $S$ . It suffices to prove that  $g = f^p$  is a non-trivial GARI of  $S$  for some positive integer  $p$ . Assume that  $(c_1, c_2, \dots, c_m) \in S^m$  such that  $f$  is defined at  $(c_1, c_2, \dots, c_m)$ . Then there exists  $p > 0$  such that  $f(c_1, c_2, \dots, c_m)^p \in F$ . Put  $g = f^p$ . It is clear that  $g$  is a GARI of  $S$  and  $g(c_1, c_2, \dots, c_m) \in F$ . One has  $g \notin D$  (since  $f$  is non-trivial GPCRI of  $S$ ), so that  $g$  is a non-trivial GARI of  $S$ .  $\square$

Let  $D$  be a division ring with center  $F$  and  $S$  be a subset of  $D^*$ . We say that  $S$  satisfies a non-trivial GARI of bounded degree if there exists a non-trivial GARI  $f(x_1, x_2, \dots, x_m)$  of  $S$  over  $D$  such that for all  $(c_1, \dots, c_m) \in S^m$ ,  $f(c_1, \dots, c_m)$  are algebraic over  $F$  of bounded degree whenever  $f(c_1, \dots, c_m)$  are defined.

**Corollary 7.** *Let  $D$  be a division ring with center  $F$  and  $f(x_1, x_2, \dots, x_m)$  be a generalized rational polynomial over  $D$ . Assume that  $S$  is a subset of  $D$  such that  $f$  is defined at least at an  $m$ -tuple  $(c_1, c_2, \dots, c_m) \in S^m$ . If  $f$  is a non-trivial GRI of  $S$ , then  $f$  is a non-trivial GARI of  $S$  of degree 1.*

*Proof.* Assume that  $f$  is a non-trivial GRI of  $S$ . Then,  $f(c_1, \dots, c_m) = 0$ , where  $(c_1, \dots, c_m) \in S^m$  such that  $f$  is defined at  $(c_1, c_2, \dots, c_m)$ . If  $f \in D$ , then  $f = 0$  that is impossible since  $f$  is non-trivial GRI of  $S$ . Hence  $f \notin D$ . In view of Theorem 5, it follows that  $f$  is a non-trivial GARI of  $S$  of degree 1.  $\square$

Now, we are ready to get the important theorem in this note.

**Theorem 8.** *Let  $D$  be a division ring with infinite center  $F$  and assume that  $N$  is a non-central subnormal subgroup of  $D^*$ . If  $N$  satisfies a non-trivial GARI of bounded degree  $d$ , then  $D$  is centrally finite, i.e.  $D$  is a finite-dimensional vector space over  $F$ .*

*Proof.* Assume that  $N$  satisfies the non-trivial GARI  $f(x_1, x_2, \dots, x_m)$  of degree  $\leq d$ . Consider

$$g_d(x, y_1, y_2, \dots, y_d) = \sum_{\sigma \in S_{d+1}} \text{sign}(\sigma) \cdot x^{\sigma(0)} y_1 x^{\sigma(1)} y_2 x^{\sigma(2)} \dots y_d x^{\sigma(d)}$$

as in Lemma 1, and put

$$w(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_d) = g_d(f(x_1, x_2, \dots, x_m), y_1, y_2, \dots, y_d).$$

Assume that  $(c_i) \in N^m$  such that  $f$  is defined at  $(c_i)$ . Since  $f(c_1, c_2, \dots, c_m)$  is algebraic over  $F$  of degree  $\leq d$ , by Lemma 1,

$$w(c_1, c_2, \dots, c_m, r_1, r_2, \dots, r_d) = 0$$

for every  $(r_i) \in D^d$ . In particular,  $w = 0$  is a GRI of  $N$ . Since  $f$  is a non-trivial GARI, there exists a division ring  $D_1$  with center  $F$  such that  $D_1$  contains all coefficients of  $f$  and  $f$  is not a GARI of  $D_1$ . By Lemma 1,  $w \neq 0$ , which implies that  $w = 0$  is a non-trivial GRI of  $N$ . Now in view of [8, Theorem 1],  $D$  is centrally finite. Thus, the proof of Theorem 8 is now complete.  $\square$

In view of corollaries 6 and 7, one can see that the result of Theorem 8 is more general than the results obtained in [8, Theorem 1] and [17].

**Remark 1.** 1) The non-triviality of  $f$  in Theorem 8 is essential. For instance, let  $D$  be a centrally infinite division ring, and assume that  $a \in D^*$  is an algebraic element of degree  $\leq d$  over the center  $F$  of  $D$ . Then, for any  $b \in D^*$ , the element  $bab^{-1}$  is always algebraic over  $F$  of degree  $\leq d$ . This means that  $f(x) = xax^{-1}$  is a GARI of  $D^*$  of degree  $d$  while  $[D : F] = \infty$ .

2) Theorem 8 does not give any evaluation of  $\dim_F D$  in terms of  $d$  yet. This problem seems to be quite interesting and remains still open in general even in the case when the GARI is a GPCRI [11, Page 137]. However, this estimation was obtained for some particular cases of GARIs of bounded degree over  $D^*$ . For instance, in [1, Theorem 4], it was proved that if all additive commutators  $xy - yx$  ( $x, y \in D^*$ ) are algebraic over  $F$  of bounded degree  $d$ , then  $\dim_F D \leq \frac{(1+d)^4}{4}$ . Later, in [7, Theorem 3], a better evaluation  $\dim_F D \leq d^2$  was obtained. Also in [7], there are analogue results for multiplicative commutators in a division ring. Namely, if there exists an integer  $d$  such that  $\dim_F F(xy x^{-1} y^{-1}) \leq d$  for all  $x, y \in D^*$ , then  $\dim_F D \leq d^2$  [7, Theorem 6]. In the case when  $\text{char}(D) = 0$ , the result is even stronger, stating that if there exists a non-central element  $a \in D$  and an integer  $d$  such that  $\dim_F F(axa^{-1}x^{-1}) \leq d$ , then  $\dim_F D \leq d^2$  [7, Theorem 7]. Skipping the evaluation of  $\dim_F D$  in terms of  $d$ , we shall extend partially the above results for a non-central subnormal subgroup  $N$  of  $D^*$ .

The following corollary is a broad extension of [18, Corollary 3] and the Jacobson Theorem [13, Theorem 7].

**Corollary 9.** *Let  $D$  be a division ring with center  $F$  and assume that  $N$  is a non-central subnormal subgroup of  $D^*$ . If all elements of  $N$  are algebraic over  $F$  of bounded degree  $d$ , then  $D$  is centrally finite. Moreover, under the additional condition  $\text{char}(D) = 0$  and  $N$  is normal in  $D^*$ , we have  $\dim_F D \leq d^2$ .*

*Proof.* If the center  $F$  is finite then every element of  $N$  is torsion. By [12, Theorem 8],  $N$  is central which contradicts the hypothesis. Hence,  $F$  is infinite. By Theorem 8,  $D$  is centrally finite.

Assume that  $\text{char}(D) = 0$  and  $N$  is normal in  $D^*$ . Now let  $a \in N \setminus F$ . For any  $x \in D^*$ ,  $axa^{-1}x^{-1} \in N$  is algebraic over  $F$  of degree  $\leq d$ . By [7, Theorem 7],  $\dim_F D \leq d^2$ . The corollary is completed.  $\square$

The following result extends partially [1, Theorem 4].

**Corollary 10.** *Let  $D$  be a division ring with infinite center  $F$  and assume that  $N$  is a non-central subnormal subgroup of  $D^*$ . If  $xy - yx$  are algebraic over  $F$  of bounded degree for all  $x, y \in N$ , then  $D$  is centrally finite.*

*Proof.* The corollary is from directly Theorem 8.  $\square$

The next two corollaries are partially generalizations of [7, Theorems 6 and 7].

**Corollary 11.** *Let  $D$  be a division ring with center  $F$  and assume that  $N$  is a non-central subnormal subgroup of  $D^*$ . If  $\text{char}(D) = 0$  and there exists an element  $a \notin F$  such that  $axa^{-1}x^{-1}$  are algebraic over  $F$  of bounded degree for all  $x \in N$ , then  $D$  is centrally finite.*

*Proof.* Put  $w(x) = axa^{-1}x^{-1}$ . Then  $w$  is a GARI of  $N$  of bounded degree. Using Theorem 5,  $w$  is a non-trivial GARI of  $N$  because  $w(1) = 1 \in F$ . Since  $F$  is infinite, by Theorem 8,  $D$  is centrally finite.  $\square$

**Corollary 12.** *Let  $D$  be a division ring with center  $F$  and assume that  $N$  is a non-central subnormal subgroup of  $D^*$ . If  $xyx^{-1}y^{-1}$  is algebraic over  $F$  of bounded degree for any  $x, y \in N$ , then  $D$  is centrally finite.*

*Proof.* Put  $w(x, y) = xyx^{-1}y^{-1}$ . Then  $w$  is a GARI of  $N$  of bounded degree. Using Theorem 5,  $w$  is a non-trivial GARI of  $N$  because  $w(1, 1) = 1 \in F$ . By Theorem 8, it suffices to show  $F$  is infinite. Indeed, assume that  $F$  is finite. Then, for any  $a, b \in N$ , the subfield  $F(aba^{-1}b^{-1})$  of  $D$  generated by  $aba^{-1}b^{-1}$  over  $F$  is finite, which implies that  $aba^{-1}b^{-1}$  is torsion of order  $n \leq |F|^d - 1$ . Here,  $|F|$  is the cardinality of  $F$ . Therefore,  $N$  satisfies a generalized group identity  $w(x, y)^n = 1$ . If there exists  $a, b \in N$  such that  $aba^{-1}b^{-1} = w(a, b) \notin F$ , then there exists a division subring  $D_1$  of  $D$  with center  $F_1$  satisfying  $D_1$  is centrally finite and  $aba^{-1}b^{-1} \notin F_1$  ([6, Proposition 2.1]). Because  $N$  is subnormal in  $D^*$ , there exist subgroups  $N_1, N_2, \dots, N_r$  of  $D^*$  such that

$$N = N_r \triangleleft N_{r-1} \triangleleft \dots \triangleleft N_1 = D^*.$$

Put  $H_i = D_1 \cap N_i$ . Then we obtain that

$$H_r \triangleleft H_{r-1} \triangleleft \dots \triangleleft H_1 = D_1^*.$$

It implies that  $H_r$  is a subnormal subgroup of  $D_1^*$  and  $aba^{-1}b^{-1} \in H_r$ . Observe that  $F_1$  is infinite and since  $H_r \subseteq N$  satisfies the group identities  $w(x, y)^n = 1$ , by [4, Theorem 3.1],  $H_r \subseteq F_1$ . In particular,  $aba^{-1}b^{-1} \in F_1$ . Contradiction! Hence,  $aba^{-1}b^{-1} \in F$  for any  $a, b \in N$ , which implies that  $N$  is solvable. In the view of [20, 14.4.4],  $N$  is central, which contradicts to the hypothesis. Thus,  $F$  is infinite and the corollary is proved completely.  $\square$

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